Linear search by a pair of distinct-speed robots

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Abstract

Two mobile robots are initially placed at the same point on an infinite line. Each robot may move on the line in either direction not exceeding its maximal speed. The robots need to find a stationary target placed at an unknown location on the line. The search is completed when both robots arrive at the target point. The target is discovered at the moment when either robot arrives at its position. The robot knowing the placement of the target may communicate it to the other robot. We look for the algorithm with the shortest possible search time (i.e. the worst-case time at which both robots meet at the target) measured as a function of the target distance from the origin (i.e. the time required to travel directly from the starting point to the target at unit velocity).

We consider two standard models of communication between the robots, namely *wireless communication* and *communication by meeting*. In the case of communication by meeting, a robot learns about the target while sharing the same location with the robot possessing this knowledge. We propose here an optimal search strategy for two robots including the respective lower bound argument, for the full spectrum of their maximal speeds. This extends the main result from [11] referring to the exact complexity of the problem for the case when the speed of the slower robot is at least one third of the faster one. In addition, we consider also the wireless communication model, in which a message sent by one robot is instantly received by the other robot, regardless of their current positions on the line. In this model, we design an optimal strategy whenever the faster robot is at most 6 times faster than the slower one.

1998 ACM Subject Classification F.2.2 Nonnumerical Algorithms and Problems

Keywords and phrases Linear search, Mobile robots, Group search, Different speeds

Digital Object Identifier 10.4230/LIPIcs.xxx.yyy.p

* Partially funded by NSERC. Part of this work was done while Jurek Czyzowicz was visiting the LaBRI as a guest professor of the University of Bordeaux.

† This work is sponsored in part by the University of Liverpool initiative Networks Systems and Technologies NeST.

‡ This work was partially funded by the ANR project MACARON (ANR-13-JS02-002). This study has been carried out in the frame of the "Investments for the future" Programme IdEx Bordeaux – CPU (ANR-10-IDEX-03-02).

§ This work was partially funded by the ANR project DISPLEXITY (ANR-11-BS02-014).

¶ This work was partially funded by the National Science Centre, Poland - grant number 2015/17/B/ST6/01897.

Conference title on which this volume is based on.
Editors: Billy Editor and Bill Editors; pp. 1–20
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

1 Introduction

Searching is a well-studied problem in which mobile robots need to find a specific target placed at some a priori unknown location. In some cases, a team of robots is involved, trying to coordinate their efforts in order to minimize the time. The complexity of the multi-robot searching is usually defined as the time when the first searcher arrives at the target position whose location is controlled by an adversary.

In distributed computing, one of the central problems is *rendezvous* when two mobile robots collaborate in order to meet in the smallest possible time. The efficiency of the rendezvous strategy is expressed as the time when the last involved robot reaches the meeting point, and the meeting point is arbitrary, i.e., the robots may choose the most convenient one.

In the *linear search* problem studied in the present paper, a pair of robots has to meet at an unknown fixed target point of the environment and the time complexity of the process is determined by the arrival of the second robot. More specifically we consider two mobile robots placed at the origin of an infinite line. Each robot has its maximal speed that it cannot exceed while moving in either direction along the line. There is a stationary target, placed at an unknown point of the line, that a robot discovers when arriving at its placement. The robot which possesses the knowledge of the target position may communicate it to the other robot. We consider two communication models of the robots: *communication by meeting* when the robots can exchange information only while being located at the same position, and *wireless communication* when the robot finding the target may instantaneously inform the other robot of its position. We want to schedule the movement of both robots so that eventually each of them arrives at the target location. The cost of the schedule is the first time when both robots are present at the target position. We express it as a function of the distance between the target and the origin.

1.1 Related work

Numerous papers have been written on the searching problem, studying diverse models involving stationary or mobile targets, graph or geometric terrain, known or unknown environment, one or many searchers, etc. (cf. [1, 3, 17, 21]). Depending on the setting, the problem is known under the name of treasure hunting, pursuit-evasion, cops and robbers, fugitive search games, etc. Sometimes the searching robot is not looking for an individual target point, attempting rather to evacuate being lost in an unknown environment or determine its position within a known map (e.g. [12, 15]). Several of these research papers offer exciting challenges of combinatorial or algorithmic nature (see [17]). In most papers studying algorithmic issues, the objective is either to determine the feasibility of the search, (i.e., whether the search will succeed under all adversarial choices) or to minimize its cost represented by the search time, assuming some given speeds of searchers (and perhaps evaders).

Most of the time searching is considered for a single robot. As one robot usually cannot map the graph being explored (unless e.g., leaving pebbles at some nodes; see [8]), the second searcher makes the task feasible (cf. [9]). However, optimization of the search by the use of multiple robots often involves coordination issues, where the searchers need to communicate in order to synchronize their efforts and adequately split the entire task into portions assigned to individual robots (cf. [11, 14, 16, 18]). As this objective is often not easy to achieve, some multi-robot search problems turn out to be NP-hard (e.g., see [18]).

Several papers on searching consider online algorithms (cf. [19]), where the information about the environment is acquired as the search progresses. The performance of an online algorithm is measured by its *competitive ratio*, i.e., the worst-case ratio of its cost with respect to the offline cost, which is the search time of the optimal algorithm with full *a priori* knowledge of the environment and the target placement. Many search problems, especially for geometric environments, are analyzed from this perspective, in particular when the cost of the offline solution is just the distance to the target; see [3, 11, 16, 19].

The *linear search* problem for a single robot was introduced by Beck [6] and Bellman [7]. They proposed an optimal on-line algorithm with search time $9d$, where d is the distance from the origin to the target. This question was extended to the *cow-path problem* in [2], in which the searcher has more than two directions to follow, to searching in the plane [3], and numerous other variations. Bose et al. [10] recently studied a variant of these problems where upper and lower bounds on the distance to the target are given. On a line, without this information the time $9d$ cannot be improved even if the search is performed by a team of same-speed robots communicating by meeting if all robots have to reach the target [11]; see also [4]. Surprisingly, time $9d$ can still be achieved by distinct-speed robots if the slowest robot is at most 3 times slower than the fastest one.

The *rendezvous problem* has been central to distributed computing for many years. It was studied in various settings (cf. [22]), but even for environments as simple as a line or a ring, optimal solutions are not always known. Feasibility of the rendezvous problem is often determined by a symmetry breaking process, which must prevent the robots from falling into an infinite pattern avoiding the meeting. Searching and rendezvous may be viewed as problems with opposite objectives. Searching is a game between a searcher, who tries to find the target as fast as possible and the adversary, who knows the searching strategy and attempts to maximize the search time by its choice of the environment parameters, target placement (or its escape route), etc. Hence in searching, the two players have contradictory goals. In rendezvous the two players collaborate, trying to quickly find one another (see [1]). Contrary to the searching problem, the rendezvous destination is not given in advance but it may be decided by the robots.

Equivalent to our setting are *evacuation problems*, where a collection of mobile robots need to find an unknown exit in the environment and the exit must be reached by all involved robots. In previous research usually robots travelling at the same speed were considered (cf. [11, 12]). For other problems considering robots with distinct speeds (e.g., the patrolling problem studied in [13, 20]), only partial results were obtained. Optimal patrolling using more than two robots on a ring [13], or more than three robots on a segment [20], is unknown in general and all intuitive solutions have been proved sub-optimal for some configurations of the speeds of the robots. Another example is the long-standing *lonely runner* conjecture [23], concerning k entities moving with constant speeds around a circular track of unit-length. If the speeds are pairwise different, the conjecture states that at some moment in time all runners are located equidistantly on the cycle. The conjecture is open in general, having been verified for up to 7 runners [5].

1.2 Our results

In this paper, we consider the linear search problem for two robots equipped with distinct maximal speeds. For the convenience of presentation we scale their speeds so that the speed of the faster robot is 1 and the slower one is $0 < v \leq 1$.

In the model with communication by meeting, we propose an optimal strategy for any value of v. And in particular our strategy works in time $\frac{1+3v}{v-v^2}d$, for any $v \leq \frac{1}{3}$ for the target being placed at unknown distance d from the origin. The remaining part of the spectrum has been covered in [11] where the authors provide: an implicit (in the limit) argument for the lower bound $9d$ when the robots share the maximal speed 1; and they show that this bound can be met from above when the slower robot's maximal speed is at least $\frac{1}{3}$.

In the model with wireless communication, we design a strategy achieving search time $\frac{2+v+\sqrt{v^2+8v}}{2v}d$. We show that this is optimal for any $v \geq \frac{1}{6}$. Note that for $v > \sqrt{17}-4 \approx 0.123$ our strategy for wireless communication outperforms the optimal strategy for communication by meeting, which shows that the feature of wireless communication is useful. On the other hand, one can observe that this feature becomes less significant as v decreases. For $v = 1$, the optimal algorithm for wireless communication is 3 times faster than the optimal algorithm for communication by meeting whereas for $v = \frac{1}{6}$, it is only 1.08 times faster.

2 Preliminaries

For any algorithm $\mathcal{A}$, we denote by $t(\mathcal{A}, p)$ the search time of algorithm $\mathcal{A}$ if the target is located at point p. We define $\tau(\mathcal{A}) = \limsup_{|p|\to\infty} \frac{t(\mathcal{A},p)}{|p|}$ as the main efficiency measure of the algorithms. Whereas all the lower bounds we derive hold for the efficiency measure $\tau(\mathcal{A})$, all the algorithms $\mathcal{A}$ we design actually satisfy the stronger property $t(\mathcal{A}, p) \leq \tau(\mathcal{A})|p|$ for every point $p \in \mathbb{R}$ (sometimes by making infinitesimal moves as the time approaches 0). In consequence, our bounds are in particular directly adaptable to a setting where the target placement must lie at a distance at least x from the origin, where x is a fixed constant, and one measures performance of the algorithms using $\sup\{\frac{t(\mathcal{A},p)}{|p|} : |p| > x\}$.

Having fixed an algorithm $\mathcal{A}$ for a set $\mathcal{R}$ of robots, each robot $\Gamma \in \mathcal{R}$ follows a fixed trajectory while it is unaware of the location of the target. We use $\Gamma(t)$ to denote the position of robot Γ at time t provided that the target location is not known to the robot. Our lower bounds rely on the analysis of the *progress speeds* $\limsup_{t\to\infty} \frac{|\Gamma(t)|}{t}$. The largest of these values over $\Gamma \in \mathcal{R}$ is called the *overall progress speed*. For each point p, the time $T(p) = \min\{t : \exists_{\Gamma\in\mathcal{R}} \Gamma(t) = p\}$ is called the *discovery time* of p (it is the first moment when any robot visits p) and $\phi(p)$ denotes the set of robots which visit p at time $T(p)$. To simplify notation, we will not make explicit the dependence of $\Gamma(t)$, $T(p)$, and $\phi(p)$ on the algorithm $\mathcal{A}$. Our results are primarily designed for a set $\mathcal{R}$ of two of robots, denoted R and r. Their speed limits are 1 and v ($v \leq 1$), respectively.

3 Communication by meeting

In this model, once a robot finds the target, it must walk to meet the other robot, and then the robots travel to the target. Naturally, the schedule consists of three phases: *exploration phase* while the target is unknown, *pursuit phase* where the informed robot chases after the other one in order to tell it about the target, and *target phase* when both robots walk to the target location. Recall that for robots with equal speeds, one of the possible optimal solutions consists in both robots following together a cow-path trajectory [4, 11], thus the pursuit and target phases may be nonexistent.

3.1 The upper bound

A robot following a standard cow-path trajectory visits, in order of increasing k, the points $p_k := (-2)^k$, $k \in \mathbb{Z}$, on alternating sides of the origin, travelling at full speed between

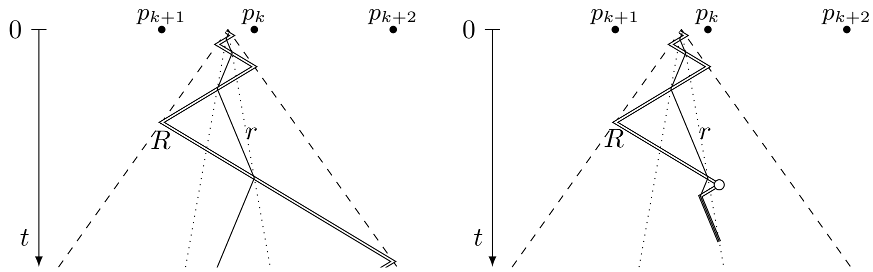


Figure 1 Illustration of algorithm $\mathcal{A}^*$ before target detection (left), and when the target has been located (right). The horizontal axis represents the line searched and the vertical axis represents the time. The empty circle denotes the target discovery. Double and single solid lines represent the trajectories of the faster and the slower robot, respectively. Dashed lines correspond to the overall progress speed and dotted lines to the search time.

consecutive points p_k.[1] In this strategy, the robot discovers new locations after it passes p_k on the way from p_{k+1} to p_{k+2}. This happens from time $t_k := |p_k| + 2\sum_{j=-\infty}^{k+1} |p_j| = 9 \cdot 2^k = 9|p_k|$ to $t'_{k+2} := |p_{k+2}| + 2\sum_{j=-\infty}^{k+1} |p_j| = 12 \cdot 2^k = 3|p_{k+2}|$. Consequently, the search time is bounded from above by $9|p|$.

As observed by Chrobak et al. [11], this strategy generalizes to a collection of two robots with speed limits 1 and $\frac{1}{3}$. Both robots follow the cow-path trajectory at their maximal speed, which means that they meet in p_k at time $t_k = 3t'_k$. When the faster robot R discovers the target at a point p between p_k and p_{k+2}, it pursues the slower robot r and brings it to the target, which turns out to be feasible within time $9|p|$; see Fig. 1.

We extend this strategy to allow $v < \frac{1}{3}$ as the speed limit of the slower robot r. We insist on the two robots meeting in points p_k at times t_k for adjusted values p_k and t_k. The smaller speed v of r allows R to travel further before going back to p_k. More formally, we increase the ratio $|p_{k+1}|/|p_k|$ and instead of taking $p_k = (-2)^k$, we set $p_k = (-c)^k$ for some $c > 2$. We still make both robots visit consecutive points p_k at their full speeds, and we choose c so that they meet in p_k while r is there for the first time and R for the second time. A condition inductively forcing the meeting at p_k to be followed by a meeting in p_{k+1} can be expressed as $\frac{1}{v}|p_{k+1} - p_k| = t_{k+1} - t_k = |p_{k+1} - p_{k+2}| + |p_{k+2} - p_k|$, i.e., $\frac{1}{v}(c+1) = 2c^2 + c - 1$. This gives $c = \frac{1+v}{2v}$, which we use for our algorithm $\mathcal{A}^*$.

The following theorem bounds the search time by robots using this strategy.

▶ **Theorem 1.** *For the algorithm $\mathcal{A}^*$ and every point $p \in \mathbb{R}$, we have:*

$$t(\mathcal{A}^*, p) = \frac{1+3v}{v-v^2}|p| \qquad \text{if } v \le \frac{1}{3}, \tag{1}$$

$$t(\mathcal{A}^*, p) = 9|p| \qquad \text{if } \frac{1}{3} < v \le 1. \tag{2}$$

Proof. First, let us show (1). Let us choose k so that the target p is located between p_k and

[1] Note that the sequence $(p_k)_{k\in\mathbb{Z}}$ is understood as prescribing infinitesimally small moves for the robot in the two directions around the origin at the beginning of the execution (when time is in the neighborhood of 0, i.e., at the beginning of the execution, the robot visits points p_k for k in the neighborhood of $-\infty$, hence it makes infinitesimal moves). Algorithm $\mathcal{A}^*$, described below, has similar behavior. In order to avoid this, we could start the sequence p_k from any finite k (instead of $-\infty$). This would result in small constant additive terms appearing throughout the calculations, but the asymptotic behavior of the algorithm and in particular the efficiency measure $\tau(\mathcal{A})$ would be unaffected.

Algorithm $\mathcal{A}^*$ [for two robots with communication by meeting]

1. Until the target is located, both robots visit, in order of increasing k, the points $p_k = (-c)^k$ for all $k \in \mathbb{Z}$, where $c = \frac{1+\tilde{v}}{2\tilde{v}}$ and $\tilde{v} = \min(v, \frac{1}{3})$. Robot R moves with speed 1 between consecutive points, and robot r with speed $\tilde{v}$.
2. When R finds the target, it moves with speed 1 to meet and notify r.
3. After the meeting, robots move together to the target at speed $\tilde{v}$.

p_{k+2}. The meeting time in p_k is $t_k = \frac{1}{v}\left(|p_k| + 2\sum_{j=-\infty}^{k-1}|p_j|\right) = \frac{1}{v}c^k\left(1 + \frac{2}{c-1}\right) = \frac{1}{v}c^k\frac{c+1}{c-1} = c^k\frac{1+3v}{v-v^2}$. Suppose that $|p - p_k| = \delta$. After meeting r in p_k, robot R needs time δ to discover the target. At that time, the distance between the robots is $\delta(1 + v)$ since they were going in opposite directions with their maximal speeds until time $t_k + \delta$. Then, the faster robot pursues the slower one. With the speed difference of $1 - v$ this takes $\frac{\delta(1+v)}{1-v}$ units of time. Next, the robots go back to the target at speed v which requires time $\frac{\delta(1+v)}{v-v^2}$, i.e., $\frac{1}{v}$ times more than the pursuit. In total, the time between t_k and the moment when both robots reach the target is $\delta + \frac{\delta(1+v)}{1-v} + \frac{\delta(1+v)}{v-v^2} = \delta\frac{v-v^2+v+v^2+1+v}{v-v^2} = \delta\frac{1+3v}{v-v^2}$. Since $t_k = |p_k|\frac{1+3v}{v-v^2}$, the total search time is $t(\mathcal{A}, p) = (|p_k| + \delta)\frac{1+3v}{v-v^2} = |p|\frac{1+3v}{v-v^2}$, as claimed.

To show (2), we simply observe that, for $v = \frac{1}{3}$, we have $\frac{1+3v}{v-v^2} = 9$. Note that for $v > \frac{1}{3}$, the searcher moving at velocity $\frac{1}{3}$ could increase its speed to v, but no additional gain in efficiency is possible (see the lower bounds in [11, 4] and in Section 3.2). ◀

3.2 The lower bound

We show that the strategy from Section 3.1 is optimal, achieving the best possible bound on the search time. In fact, some results of this section are presented in order to work for collections $\mathcal{R}$ of any number of robots. Consequently, in this section v denotes the slowest maximal speed among all the robots in $\mathcal{R}$, and r denotes some robot with maximal speed v. We also define $\tau^* = \frac{1+3v}{v-v^2}$ and, for any fixed algorithm $\mathcal{A}$, the overall progress speed $w = \max_{\Gamma\in\mathcal{R}} \limsup_{t\to\infty} \frac{|\Gamma(t)|}{t}$. Note that the functions Γ and w depend on $\mathcal{A}$, but we do not make this relation explicit in our notation.

Before we proceed with the actual lower bound, let us prove a lemma relating the search time and the overall progress speed for any collection $\mathcal{R}$ of robots.

▶ **Lemma 2.** *For any algorithm $\mathcal{A}$ and any collection $\mathcal{R}$ of robots with speeds not exceeding 1, we have $\tau(\mathcal{A}) \geq \frac{1+3w}{w-w^2}$, when $w \in (0, 1)$. If $w = 0$ or $w = 1$, then $\tau(\mathcal{A})$ cannot be bounded from above by any finite number.*

Proof. We proceed with a proof by contradiction. That is, we suppose that $\tau(\mathcal{A})$ can be bounded from above if $w \in \{0, 1\}$, and that $\tau(\mathcal{A}) < \frac{1+3w}{w-w^2}$ if $w \in (0, 1)$. In both cases, the assumption implies the existence of a finite $\bar{\tau}$ such that $\tau(\mathcal{A}) < \bar{\tau}$ and $(w - w^2)\,\bar{\tau} < 1 + 3w$. The former condition yields that there exists d_0 such that $t(\mathcal{A}, p) < \bar{\tau}|p|$ for $|p| \geq d_0$. We will obtain a contradiction with respect to the latter condition.

Let us fix $\varepsilon > 0$. Note that there exists t_0 such that $\frac{|\Gamma(t)|}{t} \leq w + \varepsilon$ for every $t \geq t_0$ and every robot $\Gamma \in \mathcal{R}$. Also, there exists a robot Γ and arbitrarily large time values t such that $\frac{|\Gamma(t)|}{t} \geq w - \varepsilon$. We fix such a robot Γ and time t, which satisfies $t \geq (\bar{\tau} - 1)\max(d_0, t_0)$.

Let $p = \Gamma(t)$ and $d_p = |p|$. Also, consider a point q at distance $d_q = \frac{t+d_p}{\bar{\tau}-1}$ from the origin on the opposite side of p; see Fig. 2. Note that $d_q \geq d_0$, so $t(\mathcal{A}, q) < \bar{\tau}d_q$.

Suppose that robot Γ at time t cannot exclude the possibility that the target is located at q. Then, it must be able to reach q by the deadline, at $t(\mathcal{A}, q) < \bar{\tau} d_q$, starting at time t from point p. The robot cannot exceed the speed limit of 1, so we conclude $\bar{\tau} d_q - t > d_p + d_q$. However, the distance d_q is defined so that $\bar{\tau} d_q - t = d_p + d_q$, a contradiction.

Consequently, robot Γ must already know at time t that the target is not at point q. Since robots can only communicate by meeting and their speeds are limited by 1, this information needs $d_q + d_p$ time to travel from q to p. In other words, some robot Γ' must have visited q at time $t' \le t - d_p - d_q$.

On the other hand, the speed limit of Γ' is at most 1, so we have $t' \ge d_q \ge t_0$. Hence, we can use a stronger bound using progress speed: $d_q = \Gamma'(t') \le t'(w + \varepsilon)$. Consequently, we obtain $d_q \le (w + \varepsilon)(t - d_p - d_q)$. Plugging in the definition of d_q, after some term rearrangements, we get $(1 + w + \varepsilon)(t + d_p) \le (t - d_p)(w + \varepsilon)(\bar{\tau} - 1)$. Equivalently, $d_p \le t\frac{(w+\varepsilon)(\bar{\tau}-2)-1}{(w+\varepsilon)\bar{\tau}+1}$. However, recall that time t was chosen so that $d_p \ge (w - \varepsilon)t$. Therefore, $w - \varepsilon \le \frac{(w+\varepsilon)(\bar{\tau}-2)-1}{(w+\varepsilon)\bar{\tau}+1}$. As $\varepsilon > 0$ can be chosen arbitrarily close to 0, we conclude that $w \le \frac{w(\bar{\tau}-2)-1}{w\bar{\tau}+1}$, that is $(w - w^2)\bar{\tau} \ge 1 + 3w$. This contradicts the definition of $\bar{\tau}$. ◀

The following immediate corollary gives an alternative proof of the optimality of $\mathcal{A}^*$ for $v \ge \frac{1}{3}$. (Recall the lower bound of 9 in [11]; see also [4].)

▶ **Corollary 3.** *For any algorithm $\mathcal{A}$ and any collection $\mathcal{R}$ of robots with speeds not exceeding 1, we have $\tau(\mathcal{A}) \ge 9$.*

Proof. It suffices to observe that $\frac{1+3w}{w-w^2} \ge 9$ for any $w \in (0, 1)$. ◀

We continue the analysis assuming that $v < \frac{1}{3}$ and $w \in (0, 1)$. We provide a series of lemmas, each imposing certain constraints on hypothetical algorithms $\mathcal{A}$ satisfying $\tau(\mathcal{A}) < \tau^*$. Eventually, we deduce that some of these constraints exclude each other. Due to space restrictions, in the main body of the paper, we only sketch most of the proofs, referring to Appendix A for full arguments, accompanied with figures illustrating them.

▶ **Lemma 4.** *If $v < \frac{1}{3}$ and $\tau(\mathcal{A}) < \tau^*$, then $w < \frac{1-v}{1+3v}$.*

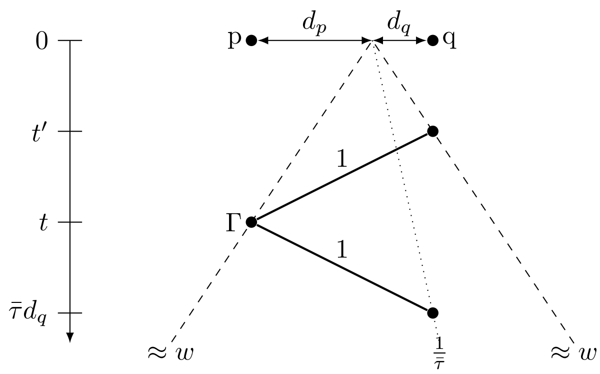


Figure 2 Illustration of notions used in the proof of Lemma 2. Rays starting from the origin as well as thick lines representing constraints are all annotated with the corresponding speeds. Here, robot Γ, while in p at time t, must know that the target is not in q, or it must be able to reach q before the deadline.

Proof. Suppose that $w \geq \frac{1-v}{1+3v}$. Note that $w \geq \frac{1-v}{1+3v} > \frac{1-\frac{1}{3}}{1+1} = \frac{1}{3}$ (because $v < \frac{1}{3}$) and the function $f(x) = \frac{1+3x}{x-x^2}$ is increasing on $(\frac{1}{3}, 1)$. Thus $\frac{1+3v}{v-v^2} = f(\frac{1-v}{1+3v}) \leq f(w) = \frac{1+3w}{w-w^2}$. Consequently, Lemma 2 implies $\tau(\mathcal{A}) \geq f(w) \geq \frac{1+3v}{v-v^2} = \tau^*$. ◀

▶ **Lemma 5.** *If $v < \frac{1}{3}$ and $\tau(\mathcal{A}) < \tau^*$, then* $\limsup_{t\to\infty} \frac{|r(t)|}{t} < \frac{vw\tau^* - wv - v - w}{vw\tau^* + 1}$.

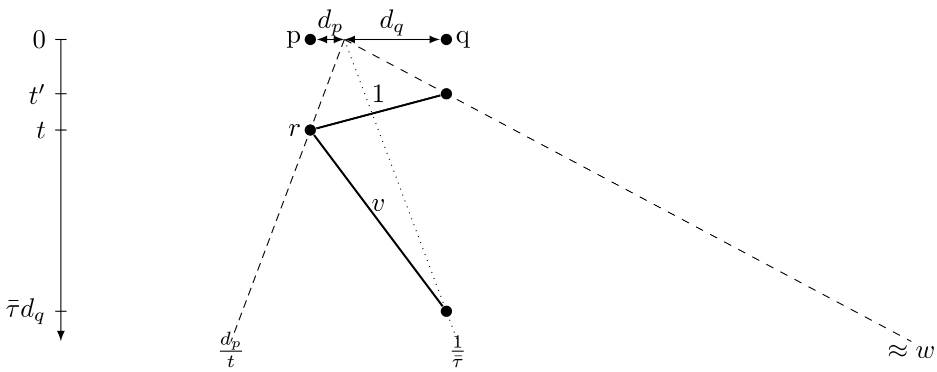


Figure 3 Illustration of notions used in the proof of Lemma 5. The slowest robot r, while in p at time t, must know that the target is not in q or it must be able to reach q before the deadline.

Proof sketch. We choose an arbitrarily large time t. Let $p = r(t)$ and $d_p = |p|$. We also consider a point q at distance $d_q = \frac{vt+d_p}{v\bar{\tau}-1}$ from the origin on the opposite side of p. Here, $\bar{\tau}$ is an arbitrary value such that $\tau(\mathcal{A}) < \bar{\tau} < \tau^*$ and $\frac{1}{v} < \bar{\tau} < \tau^* = \frac{1}{v} + \frac{4}{1-v}$. We may assume $t(\mathcal{A}, q) < \bar{\tau} d_q$ if t is chosen sufficiently large.

The distance d_q is defined so that $\frac{1}{v}(d_q + d_p) + t = \bar{\tau} d_q$. Hence, it is impossible for the slower robot r to reach point q before $\bar{\tau} d_q > t(\mathcal{A}, q)$, starting from p at time t. Consequently r already knows at time t that the target is not located at q. Hence, some robot must have visited q at time $t' \leq t - d_p - d_q$, where the inequality is due to the fact that information cannot travel faster than at speed 1. On the other hand, the progress speed w gives an upper bound on $\frac{d_q}{t'}$ as t' approaches infinity. We combine these two inequalities to bound $\frac{d_p}{t}$ from above and derive the claimed result. ◀

▶ **Corollary 6.** *If $v < \frac{1}{3}$ and $\tau(\mathcal{A}) < \tau^*$, then* $\limsup_{t\to\infty} \frac{|r(t)|}{t} < \frac{1}{\tau^*}$*. In particular, the set $\{p : r \in \phi(p)\}$ of points discovered by r is bounded.*

Proof sketch. By Lemma 4, we may assume $w < \frac{1-v}{1+3v}$. Upon substituting this inequality into the upper bound of Lemma 5, this implies $\frac{vw\tau^* - wv - v - w}{vw\tau^* + 1} < \frac{1}{\tau^*}$. Thus, the slowest robot visits sufficienly far points only after the deadline. To arrive at some location earlier, it must be notified by some other robot about the target location. ◀

While Lemmas 4 and 5 and Corollary 6 hold for arbitrary collections of robots, this is not the case for the following lemma.

▶ **Lemma 7.** *If $v < \frac{1}{3}$ and $\tau(\mathcal{A}) < \tau^*$, then* $\limsup_{t\to\infty} \frac{|r(t)|}{t} \geq \frac{vW + v + W - vW\tau^*}{vW\tau^* + 1}$ *where* $W = \frac{w-w^2}{1+3w}$.

Proof sketch. By Corollary 6, we may assume that the faster robot discovers all sufficiently far locations. Thus, its own progress speed is equal to the overall progress speed w. Moreover, the trajectory of R can be interpreted as a search algorithm for a collection $\mathcal{R} = \{R\}$ consisting of the faster robot R only. The search time of this algorithm is $T(p)$, and therefore Lemma 2 lets us conclude that $\limsup_{|p|\to\infty} \frac{T(p)}{|p|} \geq \frac{1+3w}{w-w^2} = \frac{1}{W}$.

We choose a sufficiently far point p such that $\frac{T(p)}{|p|}$ is arbitrarily close to $\frac{1}{W}$ and set $d_p = |p|$. By Corollary 6, we may assume that the slower robot does not reach p on its own before the deadline. Thus, once the faster robot discovers the target located at p, its optimal strategy is to pursue the slower robot (moving at speed 1) and then bring it to the target (moving at speed v). We define t' and $q = r(t')$ as the time and location where R catches r. The search deadline is earlier than $\tau^* d_p$, which lets us derive a lower bound on $\frac{|q|}{t'}$ and consequently bound the progress speed of the slower robot from below. ◀

▶ **Lemma 8.** *If $v < \frac{1}{3}$ and $\tau(\mathcal{A}) < \tau^*$, then $w \geq \frac{1-v}{1+3v}$.*

Proof sketch. We obtain $\frac{vw\tau^* - wv - v - w}{vw\tau^* + 1} > \frac{vW + v + W - vW\tau^*}{vW\tau^* + 1}$ using Lemmas 5 and 7. Since τ^* and W are defined using v and w only, this is an inequality on these two variables. It yields $w \geq \frac{1-v}{1+3v}$ as shown in the full proof. ◀

Lemmas 4 and 8 give conflicting constraints for any algorithm $\mathcal{A}$ such that $\tau(\mathcal{A}) < \tau^*$, which implies the following theorem.

▶ **Theorem 9.** *For any line search algorithm $\mathcal{A}$, if $v < \frac{1}{3}$, then $\tau(\mathcal{A}) \geq \tau^*$.*

4 Wireless communication

In this model, we have only the *exploration phase* and the *target phase*. We show that, for robots travelling at speeds with low relative difference (i.e., if $v \geq \frac{1}{6}$), in order to achieve the optimal search time, both robots need to participate in the exploration.

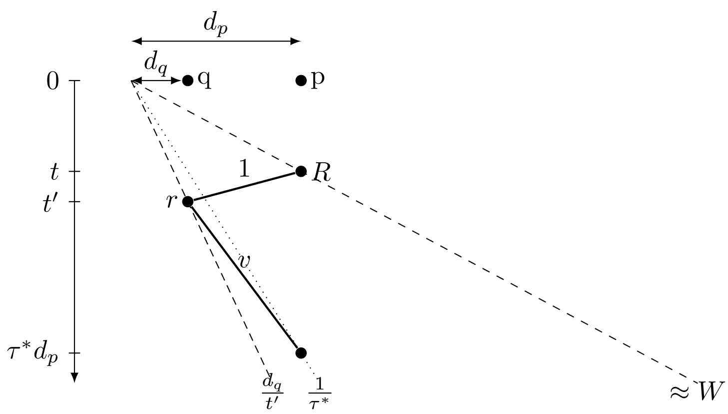


Figure 4 Illustration of notions used in the proof of Lemma 7. The faster robot R, having discovered at time t the target located at p, must be able to catch the slower robot r and bring it to the target before the deadline.

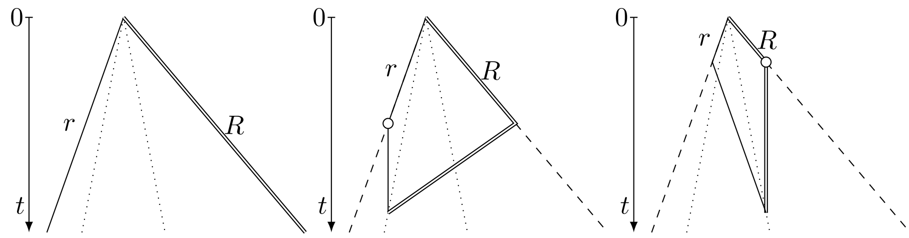


Figure 5 Illustration of algorithm $\mathcal{B}^*$ before target discovery (left), when the target is discovered by r (middle), and by R (right). The horizontal axis represents the line searched and the vertical axis represents the time. The empty circle denotes the target discovery. Double and single solid lines represent the trajectories of the faster and the slower robot, respectively. Dashed lines correspond to the progress speeds of the two robots and dotted lines to the search time.

Algorithm $\mathcal{B}^*$ [for two robots with wireless communication]

1. Until the target is discovered, the two robots move in opposite directions. Robot r moves with its maximal speed v and robot R with speed $v' = \frac{1}{2}(\sqrt{v^2 + 8v} - v)$.
2. When either robot finds the target, it notifies the other one using wireless communication and the other robot moves to the target using its maximal speed.

4.1 The upper bound

The optimal strategy for two robots travelling at the same speed [4] is very simple: Both robots explore in opposite directions at full speeds. When a robot learns that the other robot has found the target, it changes its direction towards the target.

Let us analyze the performance of this strategy for robots with distinct speeds. The total search time is a sum of three terms: the time for a robot to discover the target, the time for the other robot to go back to the origin and the time for that robot to reach the target. We consider two cases. First, suppose that the faster robot R discovers the target at distance d from the origin. Then the total search time is $d + d + \frac{1}{v}d = (2 + \frac{1}{v})d$. On the other hand, if the slower robot r discovers the target, the search time is worse: $\frac{1}{v}d + \frac{1}{v}d + d = (\frac{2}{v} + 1)d$.

Intuitively, the faster robot explores too fast and it thus spends too much time going back to the origin. Hence, we limit the exploration speed of R to $v' < 1$. When it already knows the target, the faster robot is still allowed to use its full speed equal to 1. Now, the total search times are $\frac{1}{v'}d + \frac{1}{v'}d + \frac{1}{v}d = (\frac{2}{v'} + \frac{1}{v})d$ and $\frac{1}{v}d + \frac{v'}{v}d + d = \frac{1+v'+v}{v}d$, respectively. We choose v' to minimize the maximal of these two quantities. As they are, respectively, a decreasing and an increasing function of v', for the optimal value v' these terms are equal to each other, i.e., v' satisfies $\frac{1+v'+v}{v} = \frac{2}{v'} + \frac{1}{v}$ or, equivalently, $v'^2 + v'v = 2v$.

The following fact, proved in Appendix B, gives the right values of v' and of the search time τ^*. This lets us complete the description of the algorithm $\mathcal{B}^*$ (see Fig. 5), whose analysis follows immediately from the discussion above.

▶ **Fact 10.** *For any speed $v \in (0, 1]$, define $\tau^* = \frac{2+v+\sqrt{v^2+8v}}{2v}$ and $v' = \frac{\sqrt{v^2+8v}-v}{2}$. We have: (a) $\tau^* = \frac{1+v+v'}{v}$, (b) $\tau^* = \frac{1}{v} + \frac{2}{v'}$, and (c) $v'^2 + v'v = 2v$. Moreover, if $v \geq \frac{1}{6}$, then $3v \geq v' \geq \frac{1}{2}$.*

▶ **Theorem 11.** *For the algorithm $\mathcal{B}^*$ we have $t(\mathcal{B}^*, p) = \tau^*|p|$ for every $p \in \mathbb{R}$.*

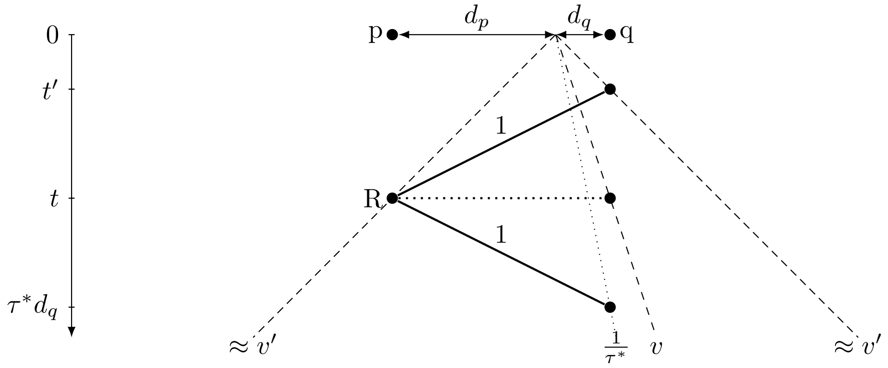


Figure 6 Illustration of notions used in the proof of Lemma 12. The faster robot R, while in p at time t, must know that the target is not in q or it must be able to reach q before the deadline. For the former, either the slower robot r must have visited q prior to t, or R must have visited q on its own and traveled all the way to p.

4.2 The lower bound

By Theorem 11, for all points p we have $t(\mathcal{B}^*, p) = \tau^*|p|$ and thus $\tau(\mathcal{B}^*) = \tau^*$. We will show that for $v \geq \frac{1}{6}$ no algorithm $\mathcal{B}$ admits a smaller value of $\tau(\mathcal{B})$. As in Section 3.2, we impose some constraints on the hypothetical algorithms, two of which are going to be inconsistent. Due to space restrictions, here we present proof sketches only; full arguments are provided in Appendix C.

▶ **Lemma 12.** *If $v \geq \frac{1}{6}$ and $\tau(\mathcal{B}) < \tau^*$, then $\limsup_{t\to\infty} \frac{|R(t)|}{t} \leq v'$.*

Proof sketch. For a proof by contradiction we suppose that the progress speed of R exceeds v'. Then, we may choose arbitrarily large time t such that $\frac{|R(t)|}{t} > v'$. Let $p = R(t)$ and $d_p = |p|$. We also consider a point q at distance $d_q = tv$ from the origin on the opposite side of p; see Fig. 6. If the time t is chosen sufficiently large, we may assume that $t(\mathcal{B}, q) < \tau^* d_q$.

The distance d_q is defined so that the robot R is unable to reach q prior to the deadline starting from p at time t. Thus, some robot must visit point q at time $t' < t$. The speed restriction for the slower robot is too strong for it to arrive at q early enough. Therefore, it must be the faster robot R which discovers q. Consequently, t' must be small enough for R to travel from q to p during time $t - t'$. On the other hand, the progress speed gives a lower bound on t' as t approaches infinity. We combine these two bounds to derive a contradiction. ◀

▶ **Lemma 13.** *If $v \geq \frac{1}{6}$ and $\tau(\mathcal{B}) < \tau^*$, then the set $\{p : r \in \phi(p)\}$ of points discovered by the slower robot r is bounded.*

Proof sketch. For a proof by contradiction, we suppose that there are arbitrarily far points discovered by robot r. We choose such a point p at distance $d_p = |p|$ from the origin. We also consider a point q at distance $d_q = \frac{2d_p}{\bar{\tau}v-1}$ from the origin on the opposite side of p. Let $\bar{\tau}$ be an arbitrary value such that $\tau(\mathcal{B}) < \bar{\tau} < \tau^*$ and $\frac{1}{v} < \bar{\tau} < \tau^* = \frac{1}{v} + \frac{2}{v'}$. We may assume $t(\mathcal{B}, p) < \bar{\tau} d_p$ and $t(\mathcal{B}, q) < \bar{\tau} d_q$ if p is chosen sufficiently far.

We analyze the discovery times $t = T(p)$ and $t' = T(q)$, and distinguish two cases depending on which is smaller. If $t \leq t'$, then robot r, while in p at time t, must be able to reach q before the deadline. The distance d_q is defined so that it is unable to do so if $t \geq \frac{d_p}{v}$, and the latter inequality easily follows from the speed limit of the slower robot r.

On the other hand, if $t' \leq t$, then the robot which visits q at time t' must be able to reach p before $\bar{\tau} d_p$. This gives an upper bound on $t' \leq \bar{\tau} d_q - d_q - d_p$ due to the speed limits. Combined with the bound of Lemma 12 on the progress speed, this yields a contradiction if the initial point p is chosen sufficiently far. ◀

▶ **Lemma 14.** *If $v \geq \frac{1}{6}$ and $\tau(\mathcal{B}) < \tau^*$, then the set $\{p : r \in \phi(p)\}$ is unbounded.*

Proof sketch. For a proof by contradiction, we suppose that the set is bounded. Then, the faster robot cannot go to infinity in one direction only, and it must pass the origin at arbitrarily large moments of time. Let us fix a sufficiently large t such that $R(t) = 0$. Consider two points p_l and p_r at distance $d = \frac{tv}{\bar{\tau} v - 1 - v}$ on each side of the origin. Let $\bar{\tau}$ be an arbitrary value such that $\tau(\mathcal{B}) < \bar{\tau} < \tau^*$ and $1 + \frac{1}{v} < \bar{\tau} < \tau^* = 1 + \frac{1}{v} + \frac{v'}{v}$. If t is chosen large enough, we may assume that $t(\mathcal{B}, p_l) < \bar{\tau} d$ and $t(\mathcal{B}, p_r) < \bar{\tau} d$ and that both points must be discovered by R.

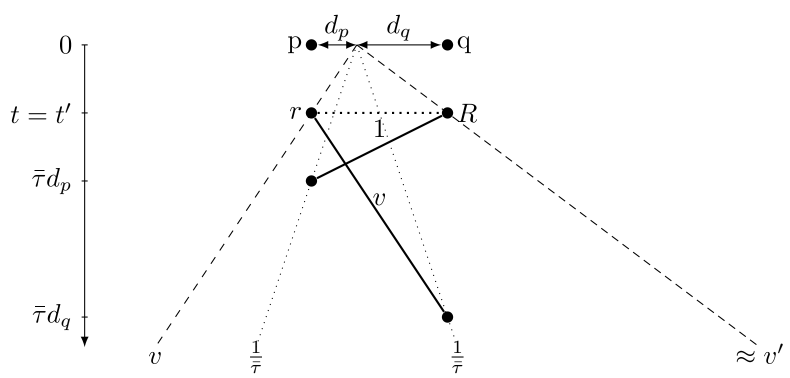
Since $R(t) = 0$, points p_l and p_r can only be discovered no later than at time $t - d$ or no sooner than at time $t + d$. The distance d is defined so that the slower robot r starting from any position at time $t + d$ is either unable to reach p_l or unable to reach p_r. Hence, one of these points must be discovered (by R) at time $t' \leq t - d$. For sufficiently large t, this contradicts the bound of Lemma 12 on the progress speed. ◀

When combined, Lemmas 13 and 14 exclude any algorithm with $\tau(\mathcal{B}) < \tau^*$.

▶ **Theorem 15.** *For any line search algorithm $\mathcal{B}$, if $\frac{1}{6} \leq v \leq 1$, then $\tau(\mathcal{B}) \geq \tau^*$.*

5 Conclusions and open questions

Clearly, any search strategy for the communication by meeting model may also be used in the wireless communication model. The bound of $\frac{1+3v}{v-v^2}$ obtained in Section 3.1 outperforms $\frac{2+v+\sqrt{v^2+8v}}{2v}$ from Section 4.1 for small values of v. More precisely, an interested reader



■ **Figure 7** Illustration of notions used in the proof of Lemma 13. The slower robot r, while discovering p at time t, must know that the target is not in q or it must be able to reach q before the deadline. For the former, the robot discovering q at t' prior to t, must be able to reach p before the deadline.

may observe that for $v \leq \sqrt{17} - 4 \approx 0.123$ we have $\frac{1+3v}{v-v^2} \leq \frac{2+v+\sqrt{v^2+8v}}{2v}$. This immediately shows that our strategy from Section 4.1 is not optimal in general. We conjecture that for $v \leq \sqrt{17} - 4$ some variation of the strategy from Section 3.1, when the faster robot is the only one responsible for exploration, will be optimal also for the wireless communication model. (Note that for general target points, it is not possible to improve the performance of the algorithm from Section 3.1 for wireless communication just by making the slower robot change direction immediately once the target is discovered by the faster robot.) As both strategies are fundamentally different, it would also be interesting to see what happens for the speeds $\sqrt{17} - 4 < v < \frac{1}{6}$.

The above fact may be viewed from another, perhaps more interesting perspective. Two unit-speed robots perform linear search in $9d$ time when communicating by meeting and in $3d$ time for the less restrictive wireless communication. Is it true that, for significantly different robot speeds, the wireless communication model loses its advantage over the communication by meeting model, and the linear search takes the same time in both models?

Another possible area of research is to extend the considerations to a larger collection of distinct-speed robots for both communication models.

References

1 Steve Alpern and Shmuel Gal. *The theory of search games and rendezvous*, volume 55 of *International Series in Operations Research & Management Science*. Kluwer Academic, 2002.

2 Ricardo A. Baeza-Yates, Joseph C. Culberson, and Gregory J. E. Rawlins. Searching with uncertainty (extended abstract). In Rolf G. Karlsson and Andrzej Lingas, editors, *Scandinavian Workshop on Algorithm Theory, SWAT 1988*, volume 318 of *LNCS*, pages 176–189. Springer, 1988.

3 Ricardo A. Baeza-Yates, Joseph C. Culberson, and Gregory J. E. Rawlins. Searching in the plane. *Information and Computation*, 106(2):234–252, 1993.

4 Ricardo A. Baeza-Yates and René Schott. Parallel searching in the plane. *Computational Geometry*, 5:143–154, 1995.

5 Javier Barajas and Oriol Serra. The lonely runner with seven runners. *Electronic Journal of Combinatorics*, 15(1), 2008.

6 Anatole Beck. On the linear search problem. *Israel Journal of Mathematics*, 2(4):221–228, 1964.

7 Richard Bellman. An optimal search. *SIAM Review*, 5(3):274–274, 1963.

8 Michael A. Bender, Antonio Fernández, Dana Ron, Amit Sahai, and Salil P. Vadhan. The power of a pebble: Exploring and mapping directed graphs. *Information and Computation*, 176(1):1–21, 2002.

9 Michael A. Bender and Donna K. Slonim. The power of team exploration: Two robots can learn unlabeled directed graphs. In *35th IEEE Annual Symposium on Foundations of Computer Science, FOCS 1994*, pages 75–85. IEEE Computer Society, 1994.

10 Prosenjit Bose, Jean-Lou De Carufel, and Stephane Durocher. Searching on a line: A complete characterization of the optimal solution. *Theoretical Computer Science*, 569:24–42, 2015.

11 Marek Chrobak, Leszek Gąsieniec, Thomas Gorry, and Russell Martin. Group search on the line. In Giuseppe F. Italiano, Tiziana Margaria-Steffen, Jaroslav Pokorný, Jean-Jacques Quisquater, and Roger Wattenhofer, editors, *Current Trends in Theory and Practice of Computer Science, SOFSEM 2015*, volume 8939 of *LNCS*, pages 164–176. Springer, 2015.

12 Jurek Czyzowicz, Leszek Gąsieniec, Thomas Gorry, Evangelos Kranakis, Russell Martin, and Dominik Pająk. Evacuating robots via unknown exit in a disk. In Fabian Kuhn, editor, *Distributed Computing, DISC 2014*, volume 8784 of *LNCS*, pages 122–136. Springer, 2014.

13 Jurek Czyzowicz, Leszek Gąsieniec, Adrian Kosowski, and Evangelos Kranakis. Boundary patrolling by mobile agents with distinct maximal speeds. In Camil Demetrescu and Magnús M. Halldórsson, editors, *Algorithms, ESA 2011*, volume 6942 of *LNCS*, pages 701–712. Springer, 2011.

14 Dariusz Dereniowski, Yann Disser, Adrian Kosowski, Dominik Pająk, and Przemysław Uznański. Fast collaborative graph exploration. *Information and Computation*, 243:37–49, 2015.

15 Gregory Dudek, Kathleen Romanik, and Sue Whitesides. Localizing a robot with minimum travel. *SIAM Journal on Computing*, 27(2):583–604, 1998.

16 Ofer Feinerman, Amos Korman, Zvi Lotker, and Jean-Sébastien Sereni. Collaborative search on the plane without communication. In Darek Kowalski and Alessandro Panconesi, editors, *ACM Symposium on Principles of Distributed Computing, PODC 2012*, pages 77–86. ACM, 2012.

17 Fedor V. Fomin and Dimitrios M. Thilikos. An annotated bibliography on guaranteed graph searching. *Theoretical Computer Science*, 399(3):236–245, 2008.

18 Pierre Fraigniaud, Leszek Gąsieniec, Dariusz R. Kowalski, and Andrzej Pelc. Collective tree exploration. *Networks*, 48(3):166–177, 2006.

19 Patrick Jaillet and Matthew Stafford. Online searching. *Operations Research*, 49(4):501–515, 2001.

20 Akitoshi Kawamura and Yusuke Kobayashi. Fence patrolling by mobile agents with distinct speeds. *Distributed Computing*, 28(2):147–154, 2015.

21 Paul J. Nahin. *Chases and Escapes: The Mathematics of Pursuit and Evasion*. Princeton Puzzlers. Princeton University Press, 2012.

22 Andrzej Pelc. Deterministic rendezvous in networks: A comprehensive survey. *Networks*, 59(3):331–347, 2012.

23 Jörg M. Wills. Zwei Sätze über inhomogene diophantische Approximation von Irrationalzahlen. *Monatshefte für Mathematik*, 71(3):263–269, 1967.

A Full proofs for Section 3.2

▶ **Lemma 5.** *If $v < \frac{1}{3}$ and $\tau(\mathcal{A}) < \tau^*$, then* $\limsup_{t\to\infty} \frac{|r(t)|}{t} < \frac{vw\tau^* - wv - v - w}{vw\tau^* + 1}$.

Proof. Take $\bar\tau$ such that $\tau(\mathcal{A}) < \bar\tau < \tau^*$ and $\frac{1}{v} < \bar\tau < \tau^* = \frac{1}{v} + \frac{4}{1-v}$. Then there exists d_0 such that $t(\mathcal{A}, p) < \bar\tau|p|$ if $|p| \geq d_0$. Fix $\varepsilon > 0$. There exists t_0 such that $\frac{|R(t)|}{t} \leq w + \varepsilon$ for every $t \geq t_0$. Let us choose time t so that $t \geq (\bar\tau - \frac{1}{v})\max(d_0, t_0)$.

Let $p = r(t)$ and $d_p = |p|$. Consider a point q at distance $d_q = \frac{vt + d_p}{v\bar\tau - 1}$ from the origin, on the other side of the origin than point p. Note that $d_q \geq \frac{vt}{v\bar\tau - 1} \geq d_0$, so $t(\mathcal{A}, q) < \bar\tau d_q$.

Suppose that the slowest robot r does not know at time t whether the target is in q. Then it must be able to reach q earlier than $\bar\tau d_q$ starting from p at time t. It cannot exceed the speed limit v, so we must have $(\bar\tau d_q - t)v > d_p + d_q$. However, d_q is defined so that $(\bar\tau d_q - t)v = d_p + d_q$, a contradiction.

Thus, the slowest robot r must know at time t that the target is not at point q. By the speed limit, this information needs $d_p + d_q$ time to travel from q to p. In other words, some robot must have visited q at time $t' \leq t - d_p - d_q$.

On the other hand, we have $t' \geq d_q \geq t_0$ so $d_q \leq t'(w+\varepsilon)$. Consequently, $d_q \leq (w+\varepsilon)(t-d_p-d_q)$. Taking into account the definition of d_q, after term rearrangements we get

$$(1+w+\varepsilon)(vt+d_p) \leq (t-d_p)(w+\varepsilon)(v\bar{\tau}-1).$$

Equivalently, $|r(t)| = d_p \leq t\frac{(v\bar{\tau}-v-1)(w+\varepsilon)-v}{v\bar{\tau}(w+\varepsilon)+1}$.

Since t can be any sufficiently large positive real number, we have

$$\limsup_{t\to\infty} \frac{|r(t)|}{t} \leq \frac{(v\bar{\tau}-v-1)(w+\varepsilon)-v}{v\bar{\tau}(w+\varepsilon)+1}.$$

By freedom to choose arbitrary $\varepsilon > 0$, we infer

$$\limsup_{t\to\infty} \frac{|r(t)|}{t} \leq \frac{vw\bar{\tau}-vw-w-v}{vw\bar{\tau}+1} < \frac{vw\tau^*-vw-w-v}{vw\tau^*+1}$$

where the last inequality follows from monotonicity and $\bar{\tau} < \tau^*$. ◀

▶ **Corollary 6.** *If $v < \frac{1}{3}$ and $\tau(\mathcal{A}) < \tau^*$, then $\limsup_{t\to\infty} \frac{|r(t)|}{t} < \frac{1}{\tau^*}$. In particular, the set $\{p : r \in \phi(p)\}$ of points discovered by r is bounded.*

Proof. Note that

$$f(x) = \frac{(v\tau^*-v-1)x-v}{v\tau^*x+1} = \frac{\left(\frac{1+3v}{1-v}-v-1\right)x-v}{\frac{1+3v}{1-v}x+1} = \frac{v((3+v)x-(1-v))}{(1+3v)x+(1-v)}$$

is an increasing function for $x > 0$, since $(3+v)\cdot(1-v)+(1+3v)\cdot(1-v) > 0$. By Lemma 4 we may assume $w < \frac{1-v}{1+3v}$. Combining $f(w) < f(\frac{1-v}{1+3v})$ with Lemma 5 gives

$$\limsup_{x\to\infty} \frac{|r(x)|}{x} < f(w) < f\left(\frac{1-v}{1+3v}\right) = \frac{v((3+v)\frac{1-v}{1+3v}-(1-v))}{(1+3v)\frac{1-v}{1+3v}+(1-v)} = \frac{v(\frac{3+v}{1+3v}-1)}{2} = \frac{2v-2v^2}{2(1+3v)} = \frac{1}{\tau^*}.$$

Consequently, $\limsup_{x\to\infty} \frac{|r(x)|}{x} < \frac{1}{\tau^*}$.

In particular, for any large enough $|p|$, if $r \in \phi(p)$, we would have $T(p) > \tau^*|p|$, a contradiction. (Recall that $T(p)$ denotes the discovery time of p.) ◀

▶ **Lemma 7.** *If $v < \frac{1}{3}$ and $\tau(\mathcal{A}) < \tau^*$, then $\limsup_{t\to\infty} \frac{|r(t)|}{t} \geq \frac{vW+v+W-vW\tau^*}{vW\tau^*+1}$ where $W = \frac{w-w^2}{1+3w}$.*

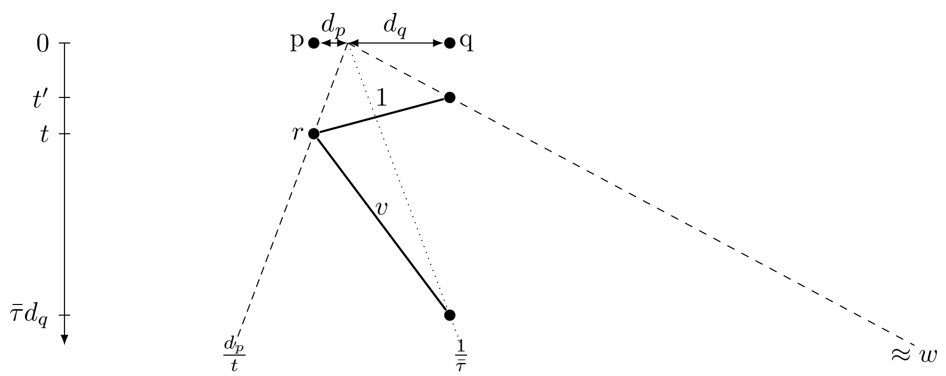


Figure 8 Illustration of notions used in the proof of Lemma 5. The slowest robot r, while in p at time t, must know that the target is not in q or it must be able to reach q before the deadline.

Proof. Recall that there exists d_0 such that $t(\mathcal{A},p) < \tau^*|p|$ if $|p| \geq d_0$. By Corollary 6, $\limsup_{t\to\infty} \frac{|r(t)|}{t} < \frac{1}{\tau^*}$, so there exists t_0 such that $\frac{|r(t)|}{t} < \frac{1}{\tau^*}$ for every $t > t_0$. Moreover, we have $y := \sup\{|p| : r \in \phi(p)\} < \infty$ and thus $w = \limsup_{t\to\infty} \frac{|R(t)|}{t}$.

Note that the trajectory of R can be interpreted as a search algorithm for a collection of $\mathcal{R} = \{R\}$ of one robot. Its search time is equal to its discovery time $T(p)$ and by Lemma 2 we can conclude that $\limsup_{|p|\to\infty} \frac{T(p)}{|p|} \geq \frac{1+3w}{w-w^2} = \frac{1}{W}$. In other words, for every $\varepsilon > 0$ there exist arbitrarily far points p such that $t := T(p) \geq \frac{|p|}{W+\varepsilon}$. We fix ε and choose p so that $d_p = |p|$ satisfies $d_p \geq \max(d_0, y)$ and $d_p \geq t_0 v$.

The slower robot has strictly less than $d_p\tau^* - t$ time to get to p from time t if the target is located there. Since $d_p > t_0 v$, the slower robot cannot visit p prior to the deadline without knowing that the target is at p. Thus, the faster robot must meet the slower one at some time t', $t < t' < d_p\tau^*$ and then the two robots shall go to the target with speed v.

Let $q = r(t')$ and let $d_q = |q|$. We have $d_p\tau^* - t' > \frac{1}{v}(d_p - d_q)$ and $t' - t \geq d_p - d_q$, i.e., $\frac{d_q}{t'} > v - \frac{d_p(v\tau^*-1)}{t'}$ and $\frac{d_q}{t'} \geq \frac{d_p+t}{t'} - 1$. Taking an appropriate convex combination of these two inequalities, we conclude

$$\frac{d_q}{t'}\left(\frac{1}{d_p(v\tau^*-1)} + \frac{1}{d_p+t}\right) \geq \frac{v}{d_p(v\tau^*-1)} - \frac{1}{d_p+t},$$

i.e.,

$$\frac{|r(t')|}{t'} = \frac{d_q}{t'} \geq \frac{v(d_p+t)-d_p(v\tau^*-1)}{d_p+t+d_p(v\tau^*-1)} = \frac{vd_p+tv+d_p-d_pv\tau^*}{d_pv\tau^*+t}.$$

The right-hand side is an increasing function of t, since $v \cdot d_pv\tau^* - (vd_p + d_p - d_pv\tau^*) = d_p(v+1)(v\tau^* - 1) > 0$. Thus, we can apply $t \geq \frac{d_p}{W+\varepsilon}$ to obtain

$$\frac{|r(t')|}{t'} \geq \frac{v(W+\varepsilon)+v+(W+\varepsilon)-v(W+\varepsilon)\tau^*}{v(W+\varepsilon)\tau^*+1}.$$

Since $t' > t$ and t can be chosen arbitrarily large, we can replace the left-hand side with $\limsup_{t'\to\infty} \frac{|r(t')|}{t'}$ and, since ε can be chosen arbitrarily small, we obtain

$$\limsup_{t\to\infty} \frac{|r(t)|}{t} \geq \frac{vW+v+W-vW\tau^*}{vW\tau^*+1}$$

as claimed. ◀

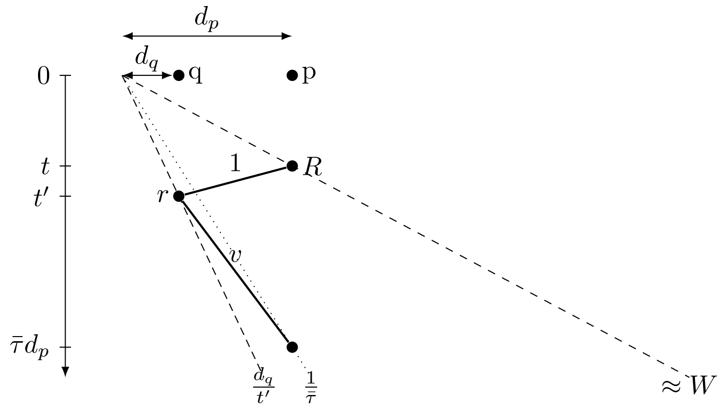


Figure 9 Illustration of notions used in the proof of Lemma 7. The faster robot R, having discovered at time t the target located at p, must be able to catch the slower robot r and bring it to the target before the deadline.

▶ **Lemma 8.** *If $v < \frac{1}{3}$ and $\tau(\mathcal{A}) < \tau^*$, then $w \geq \frac{1-v}{1+3v}$.*

Proof. Combining Lemmas 5 and 7, we get

$$\frac{vw\tau^* - wv - v - w}{vw\tau^* + 1} > \frac{vW + v + W - vW\tau^*}{vW\tau^* + 1}.$$

Expanding with the definition of τ^* and W gives

$$\frac{(3+v)w - (1-v)}{w(1+3v) + (1-v)} > \frac{1 - v - W(3+v)}{W(1+3v) + (1-v)} = \frac{1 - v - 4vw + (3+v)w^2}{1 - v + 4w - (1+3v)w^2}.$$

Note that the denominators are positive so the following inequality is equivalent.

$$((3+v)w - (1-v))(1 - v + 4w - (1+3v)w^2) >$$
$$(1 - v - 4vw + (3+v)w^2)(w(1+3v) + (1-v)).$$

The corresponding polynomial has roots at $w \in \{1, \frac{1-v}{1+3v}, -\frac{1-v}{3+v}\}$. Thus, the (strict) inequality either holds in the whole interval $[0, \frac{1-v}{1+3v})$ or in none of these points. The left-hand side has a negative coefficient with w^0 while the right-hand side has a positive one, so this inequality is false for $w = 0$, and thus for every $w \in [0, \frac{1-v}{1+3v})$. Consequently, $w \geq \frac{1-v}{1+3v}$, as claimed. ◀

B Proof of Fact 10

▶ **Fact 10.** *For any speed $v \in (0, 1]$, define $\tau^* = \frac{2+v+\sqrt{v^2+8v}}{2v}$ and $v' = \frac{\sqrt{v^2+8v}-v}{2}$. We have: (a) $\tau^* = \frac{1+v+v'}{v}$, (b) $\tau^* = \frac{1}{v} + \frac{2}{v'}$, and (c) $v'^2 + v'v = 2v$. Moreover, if $v \geq \frac{1}{6}$, then $3v \geq v' \geq \frac{1}{2}$.*

Proof. Recall that $v' = \frac{\sqrt{v^2+8v}-v}{2}$ and $\tau^* = \frac{2+v+\sqrt{v^2+8v}}{2v}$. Consequently,

$$\frac{1+v+v'}{v} = \frac{2+2v+2v'}{2v} = \frac{2+2v+\sqrt{v^2+8v}-v}{2v} = \frac{2+v+\sqrt{v^2+8v}}{2v} = \tau^*,$$

as claimed in (a). Similarly we prove (b):

$$\frac{1}{v} + \frac{2}{v'} = \frac{1}{v} + \frac{4}{\sqrt{v^2+8v}-v} = \frac{1}{v} + \frac{4(\sqrt{v^2+8v}+v)}{(\sqrt{v^2+8v}-v)(\sqrt{v^2+8v}+v)} = \frac{1}{v} + \frac{4(\sqrt{v^2+8v}+v)}{8v} = \tau^*.$$

Combining these two equalities, we get (c):

$$\frac{1+v+v'}{v} = \frac{1}{v} + \frac{2}{v'},$$
$$\frac{v+v'}{v} = \frac{2}{v'},$$
$$vv' + v'^2 = 2v.$$

Moreover,

$$\frac{v'}{v} = \frac{\sqrt{v^2+8v}-v}{2v} = \frac{4}{\sqrt{v^2+8v}+v}$$

and

$$v' = \frac{4v}{\sqrt{v^2+8v}+v} = \frac{4}{\sqrt{1+8/v}+1}.$$

Consequently, $\frac{v'}{v}$ and v' are respectively a decreasing and an increasing function of v. For $v = \frac{1}{6}$ we have $3v = v' = \frac{1}{2}$, and thus $3v \geq v' \geq \frac{1}{2}$ for every $v \geq \frac{1}{6}$. ◀

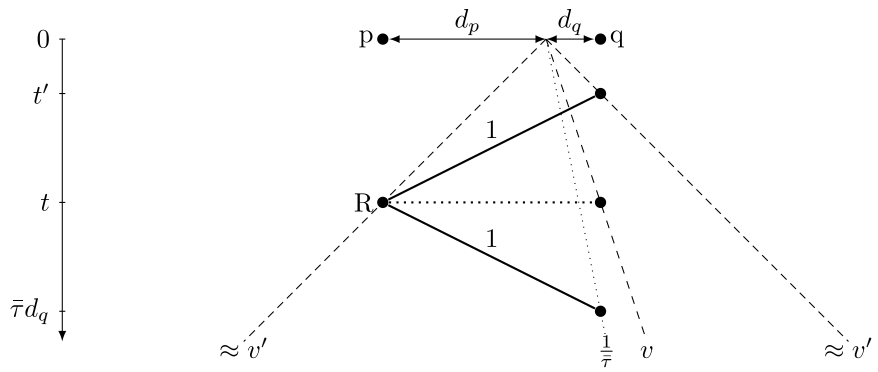


Figure 10 Illustration of notions used in the proof of Lemma 12. The faster robot R, while in p at time t, must know that the target is not in q or it must be able to reach q before the deadline. For the former, either the slower robot r must have visited q prior to t, or R must have visited q on its own and traveled all the way to p.

C Full proofs for Section 4.2

▶ **Lemma 12.** *If $v \geq \frac{1}{6}$ and $\tau(\mathcal{B}) < \tau^*$, then $\limsup_{t\to\infty} \frac{|R(t)|}{t} \leq v'$.*

Proof. Note that there exists d_0 such that $t(\mathcal{B}, p) < \tau^*|p|$ if $|p| \geq d_0$.

Let $v_{\max} = \limsup_{t\to\infty} \frac{|R(t)|}{t}$. For a proof by contradiction suppose $v_{\max} > v'$. Fix $\varepsilon > 0$ sufficiently small ($\varepsilon < v_{\max} - v'$) and note that there exists t_0 such that $\frac{|R(t)|}{t} \leq v_{\max} + \varepsilon$ for every $t \geq t_0$. Also, there exist arbitrarily large time values t such that $\frac{|R(t)|}{t} \geq v_{\max} - \varepsilon \geq v'$. We shall choose such t which satisfies $t \geq \frac{t_0}{v}$ and $t \geq \frac{d_0}{v}$.

Let $p = R(t)$ and $d_p = |p|$. Consider a point q at distance $d_q = tv$ from the origin, on the other side of the origin than point p. Note that $d_q = tv \geq d_0$, so $t(\mathcal{B}, q) < \tau^* d_q$. Suppose q was not visited by any robot prior to time t. Then the faster robot R must be able to reach q earlier than $\tau^* d_q$ starting from p at time t. As it cannot exceed the speed limit, we have $\tau^* d_q - t > d_p + d_q \geq v't + d_q$. We have $d_q = tv$, so this implies $\tau^* v - 1 > v' + v$ which is a contradiction since $\tau^* = 1 + \frac{1}{v} + \frac{v'}{v}$ by Fact 10(a).

Thus, point q must have been visited by some robot at time $t' < t$. First, observe that since v is the maximal speed of the slower robot r, point q has not been visited by r. Note that $t' \geq d_q = tv$ because the maximal speed of the faster robot is 1. Consequently, $t' \geq t_0$ and thus $d_q = |R(t')| \leq t'(v_{\max} + \varepsilon)$. Additionally, $t - t' \geq d_q + d_p$, (i.e., $t' \leq t - d_q - d_p$) by the speed limit of 1 for the faster robot when moving from q at time t' to p by time t. Combining these inequalities we get $d_q \leq (t - d_q - d_p)(v_{\max} + \varepsilon)$, and thus

$$tv = d_q \leq (t - d_q - d_p)(v_{\max} + \varepsilon) \leq (t - tv - t(v_{\max} - \varepsilon))(v_{\max} + \varepsilon),$$

i.e.,

$$v \leq (1 - v - v_{\max} + \varepsilon)(v_{\max} + \varepsilon).$$

Since $\varepsilon > 0$ could be chosen arbitrarily small, this yields

$$v \leq (1 - v - v_{\max})v_{\max}$$

Note that $f(x) = (1 - v - x)x$ is (strictly) decreasing for $x \geq \frac{1-v}{2}$, in particular for $x \geq v' \geq \frac{1}{2} \geq \frac{1-v}{2}$. This implies $v \leq f(v_{\max}) < f(v') = (1 - v - v')v' = v' - 2v \leq v$ since $v' \leq 3v$ and $v'^2 + v'v = 2v$ by Fact 10. This contradiction concludes the proof. ◀

▶ **Lemma 13.** *If $v \geq \frac{1}{6}$ and $\tau(\mathcal{B}) < \tau^*$, then the set $\{p : r \in \phi(p)\}$ of points discovered by the slower robot r is bounded.*

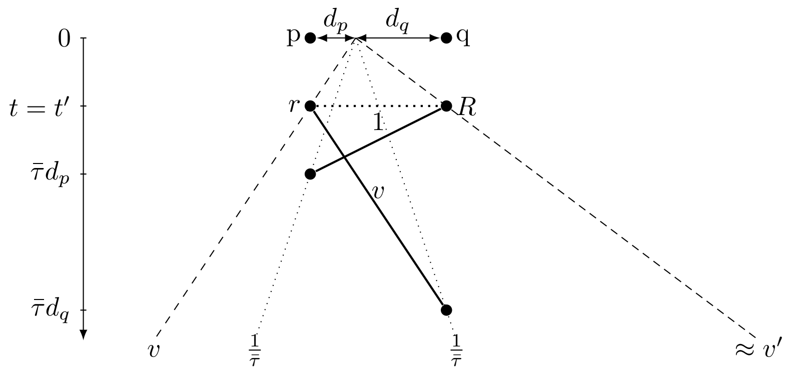


Figure 11 Illustration of notions used in the proof of Lemma 13. The slower robot r, while discovering p at time t, must know that the target is not in q or it must be able to reach q before the deadline. For the former, the robot discovering q at t' prior to t, must be able to reach p before the deadline.

Proof. Let us choose $\bar{\tau}$ such that $\tau(\mathcal{B}) < \bar{\tau} < \tau^*$ and $\frac{1}{v} < \bar{\tau} < \tau^* = \frac{1}{v} + \frac{2}{v'}$. Then there exists d_0 such that $t(\mathcal{B}, p) < \bar{\tau}|p|$ if $|p| \geq d_0$. Fix $\varepsilon > 0$. By Lemma 12 we can assume that $\limsup_{t\to\infty} \frac{|R(t)|}{t} \leq v'$, so there exists t_0 such that $\frac{|R(t)|}{t} \leq v' + \varepsilon$ for every $t \geq t_0$.

For a proof by contradiction suppose there exist points p such that $r \in \phi(p)$ and the distances $d_p = |p|$ to the origin are arbitrarily large. We choose such a point so that $d_p \geq d_0$ and $d_p \geq \max(d_0, t_0)\frac{(\bar{\tau}v-1)}{2}$.

Let q be a point at distance $d_q = \frac{2d_p}{\bar{\tau}v-1}$ from the origin on the opposite side of p. Note that $d_q \geq d_0$ so $t(\mathcal{B}, q) < \bar{\tau}d_q$. Similarly, $t(\mathcal{B}, p) < \bar{\tau}d_p$.

Let $t = T(p)$ and $t' = T(q)$. First, suppose that $t' \geq t$, i.e., q was discovered after p. Then the slower robot while in p at time t must be able to reach q before $\bar{\tau}d_q$. It cannot exceed its maximal speed v, so we must have $(\bar{\tau}d_q - t)v > d_p + d_q$, i.e., $d_q(\bar{\tau}v - 1) > d_p + tv$. By the speed limit on the slower robot we have $tv \geq d_p$, so $d_q(\bar{\tau}v - 1) > 2d_p$. However, d_q is defined so that $d_q(\bar{\tau}v - 1) = 2d_p$, a contradiction.

Consequently, $t' \leq t$, i.e., one of the robots must have visited q before p was discovered. Since p was not visited prior to time t', this robot must be able to reach p before $t(\mathcal{B}, p)$ starting from q at time t'. Recall that we have $t(\mathcal{B}, p) < \bar{\tau}d_p$. Thus, we must have $\bar{\tau}d_p - t' > d_p + d_q$ so that the robot does not exceed the maximal speed 1.

On the other hand, we have $t' \geq d_q \geq t_0$ and thus $d_q \leq (v' + \varepsilon)t'$. Combining these two inequalities, we get $\frac{d_q}{v'+\varepsilon} \leq t' < \bar{\tau}d_p - d_p - d_q$. Expanding with the definition of d_q, we obtain $\frac{2d_p}{(\bar{\tau}v-1)(v'+\varepsilon)} < (\bar{\tau} - 1)d_p - \frac{2d_p}{\bar{\tau}v-1}$, i.e., $2(1 + v' + \varepsilon) < (\bar{\tau} - 1)(\bar{\tau}v - 1)(v' + \varepsilon)$. By the freedom to choose arbitrarily small ε, we get $2(1 + v') < (\bar{\tau} - 1)(\bar{\tau}v - 1)v'$. The right-hand side is monotone with respect to $\bar{\tau}$, so

$$2(1 + v') \leq (\bar{\tau} - 1)(\bar{\tau}v - 1)v' < (\tau^* - 1)(\tau^*v - 1)v' = \frac{1+v'}{v}(v + v')v'.$$

Consequently, $2v < (v + v')v'$ which contradicts Fact 10. ◀

▶ **Lemma 14.** *If $v \geq \frac{1}{6}$ and $\tau(\mathcal{B}) < \tau^*$, then the set $\{p : r \in \phi(p)\}$ is unbounded.*

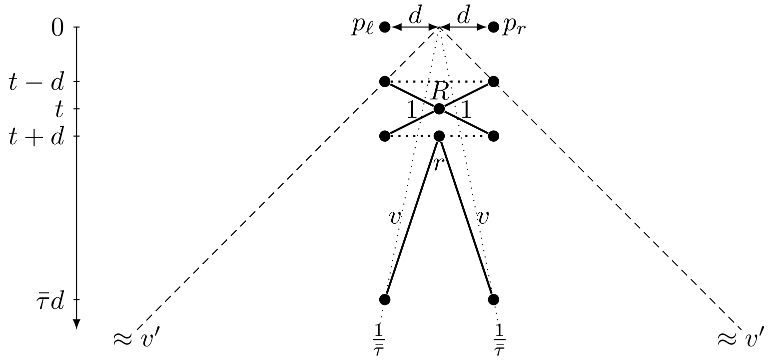


Figure 12 Illustration of notions used in the proof of Lemma 14. The slower robot r at time $t+d$ must either be able to reach both p_ℓ and p_r before the deadline, or the faster robot R must have visited one of these points prior to time $t+d$, which actually could only happen prior to time $t-d$.

Proof. Let us choose $\bar{\tau}$ such that $\tau(\mathcal{B}) < \bar{\tau} < \tau^*$ and $1 + \frac{1}{v} < \bar{\tau} < \tau^* = 1 + \frac{1}{v} + \frac{v'}{v}$. Then there exists d_0 such that $t(\mathcal{B}, p) < \bar{\tau}|p|$ if $|p| \geq d_0$. Fix $\varepsilon > 0$. By Lemma 12 we can assume that $\limsup_{t\to\infty} \frac{|R(t)|}{t} \leq v'$, so there exists t_0 such that $\frac{|R(t)|}{t} \leq v' + \varepsilon$ for every $t \geq t_0$.

For a proof by contradiction, suppose that $y = \sup\{|p| : r \in \phi(p)\} < \infty$. This means that all points farther than y from the origin are discovered by the faster robot R. Thus, R cannot go to infinity in only one direction. Hence, there exist arbitrarily large time moments t satisfying $R(t) = 0$. We shall take such t so that $\frac{tv}{\bar{\tau}v-1-v} \geq \max(d_0, y, t_0)$.

Consider two points p_l and p_r, both at distance $d = \frac{tv}{\bar{\tau}v-1-v}$ to the left and to the right from the origin, respectively. Note that $d \geq d_0$ so the target located at p_l or p_r would need to be reached earlier than $\bar{\tau}d$. First, suppose that none of these points has been discovered prior to time t.

Consider time moment $t+d$, which is the earliest moment when one of the points p_l, p_r may be discovered. At this time the slower robot must be able to reach both p_l and p_r before the time $\bar{\tau}d$. The distance between these points is $2d$, so the robot has to traverse at least distance d in time strictly less than $\bar{\tau}d-t-d$. In other words, we must have $v(\bar{\tau}d-t-d) > d$, i.e., $tv < d(\bar{\tau}v - 1 - v)$. However, $d = \frac{tv}{\bar{\tau}v-1-v}$, a contradiction.

Therefore, at least one of the points p_l, p_r must have been visited before time t. Since $d \geq y$, this point must have been discovered by the faster robot, which could do that only at time $t' \leq t-d$. On the other hand, $t' \geq d \geq t_0$, so we have $d = |R(t')| \leq (v'+\varepsilon)t' \leq (v'+\varepsilon)(t-d)$. Expanding with the definition of d, we obtain

$$\frac{tv}{\bar{\tau}v-1-v} \leq (v'+\varepsilon)\left(t - \frac{tv}{\bar{\tau}v-1-v}\right),$$

i.e.,

$$v \leq (v'+\varepsilon)(\bar{\tau}v - 1 - 2v).$$

Since the choice of ε was arbitrary, we get

$$v \leq v'(\bar{\tau}v - 1 - 2v) < v'(\tau^* v - 1 - 2v) = v'(v' - v),$$

However, by Fact 10, we have $v'(v'-v) = 2(v - vv') \leq v$. This contradiction concludes the proof. ◀